Physics 618 2020

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Comparison of two expressions for

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = Z(\phi_2, \phi_1 ; \beta)$$

Hamiltonian approach:

$$\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2I}(m-B)^2 + im(\phi_1 - \phi_2)}$$

$\boxed{\beta \to \infty}$

Path integral / S.C. Evaluation

$$\sqrt{\frac{I}{2\pi\beta}} \sum_{w \in \mathbb{Z}} e^{-\frac{2\pi^2 I}{\beta}\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)^2 + 2\pi i B\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)}$$

$\boxed{\beta \to 0}$

Mathematical Proof

Poisson Summation: for any "good" function $f: \mathbb{R} \longrightarrow \mathbb{C}$

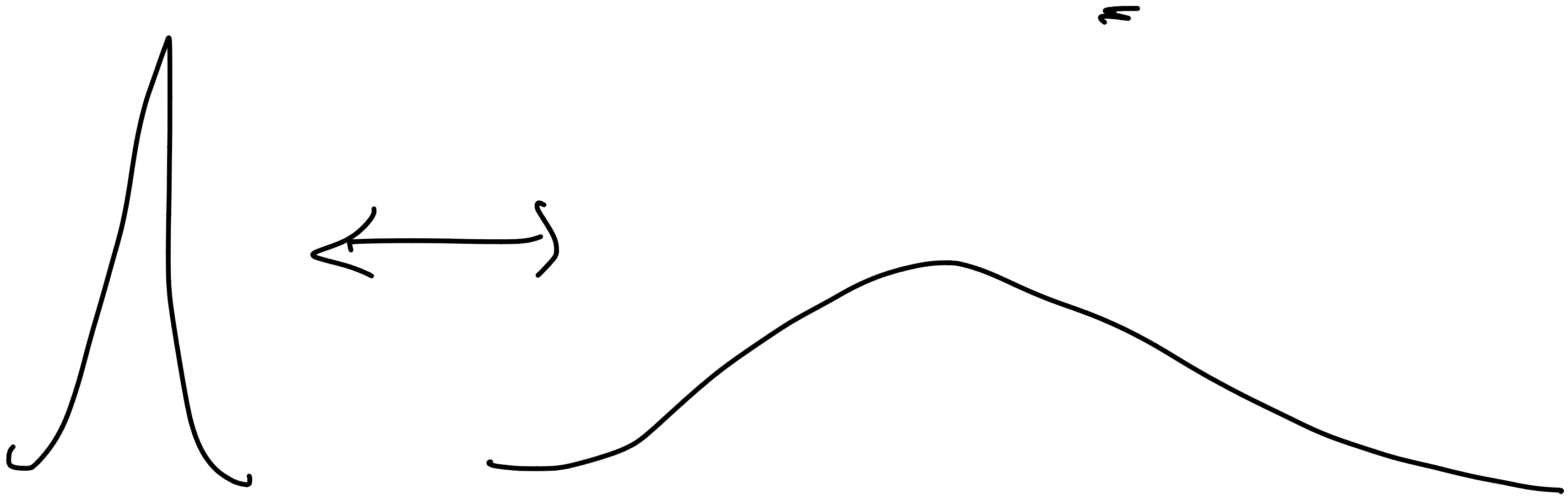
$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{w \in \mathbb{Z}} \hat{f}(w)$$

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-2\pi i w t} f(t)\, dt$$

- $f, \hat{f}$ decay so that sums make sense.

F.T. of a Gaussian = Gaussian

$$e^{-ax^2} \rightsquigarrow e^{-\frac{1}{a}x^2}$$



Riemann theta function

characteristics

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z|\tau) :=$$

$z \in$ complex torus $E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

$$\sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\theta)^2 + 2\pi i(n+\theta)(z+\phi)}$$

$\theta, \phi \in \mathbb{R}$ $\quad z \in \mathbb{C}$ $\quad \tau \in \mathbb{H}$

$\operatorname{Im}\tau > 0$

analytic function of $z \in \mathbb{C}$.

P.S.F. :

weight $= 1/2$

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left(-\frac{z}{\tau}\Big| -\frac{1}{\tau}\right) = (-i\tau)^{1/2} e^{2\pi i\theta\phi}$$

$$= e^{i\pi z^2/\tau}\,\vartheta\begin{bmatrix}-\phi\\ \theta\end{bmatrix}(z|\tau)$$

index $= 1/2$

Modular trm law of ϑ.

"Hamiltonian"

$$Z(\phi_2, \phi_1 | \beta) = \frac{1}{2\pi} e^{i\mathcal{B}(\phi_1 - \phi_2)} \vartheta\begin{bmatrix}\theta \\ \phi\end{bmatrix}(0|\tau)$$

$$\tau = i\frac{\beta}{2\pi I} \qquad \theta = -\mathcal{B} \qquad \phi = \frac{\phi_1 - \phi_2}{2\pi}$$

$$= \sqrt{\frac{I}{2\pi\beta}}\, \vartheta\begin{bmatrix}\theta' \\ \phi'\end{bmatrix}(0|\tau')$$

$$\tau' = i\frac{2\pi I}{\beta} \qquad \theta' = \frac{\phi_1 - \phi_2}{2\pi} \qquad \phi' = -\mathcal{B}$$

$$\tau' = -1/\tau$$

$\beta \to 0$ high temp

$$Z(S^1) = \int d\phi \langle \phi | e^{-\beta H} | \phi \rangle$$

inst. correcting

$$\underset{\beta \to 0}{\sim} \left(\frac{2\pi I}{\beta \hbar}\right)^{1/2} \left[1 + 2e^{-\frac{2\pi^2 I}{\beta\hbar}} \cos(2\pi\mathcal{B}) + \cdots\right]$$

$\mathbb{Z}_2 : \beta \to 1/\beta$

Other Examples of this mechanism:

1.) T-dualities of CFT's with flat toroidal target spaces. $\to O(p,q;\mathbb{Z})$

2.) S-duality in 4 dim'l Abelian gauge theories $SL(2,\mathbb{Z})$

$Sp(2g;\mathbb{Z})$

3.) p-form gauge theories....

Continue w/ this Q.M. System - Illustrate Some Ideas from gauge theory, Chern-Simons terms.

Physical system has a "global internal" symmetry (s.t. $\phi \to \phi + \alpha$

$SO(2)$ or $\Phi(t) = e^{i\phi(t)}$

$\Phi(t) \longrightarrow e^{i\alpha}\,\Phi(t)$ $U(1)$ symmetry)

"Gauge the symmetry"

2- Step process:

Heursitc:

1. Make symmetry local by coupling to a gauge field.

2. Integrate / Sum over all possible gauge fields.

Mathematically precise:

1. Change the bordism category in the domain of the field theory functor to include principal G-bundles / spacetime w/ connection.

2. Sum/Integrate over isom. classes of principal bundles with connection.

Here gauge $\boxed{\phi(t) \to \phi(t) + \underset{\text{const.}}{\alpha}}$

SO(2) symmetry.

$$\phi(t) \to -\phi(t)$$

Remark: You can stop with Step 1: Making the theory equivariant. Then the gauge fields are "external data" "background fields" i.e. we do not do a path integral over these fields.

$$\int \frac{1}{2} I \dot{\phi}^2 \, dt + \int \mathcal{B} \dot{\phi} \, dt$$

local: $\phi(t) \longrightarrow \phi(t) + \underbrace{\alpha(t)}_{\text{t-dep. parameter}}$

new data: $A^{(e)}(t)$

$$S' = \int \tfrac{1}{2} I \left(\dot\phi + A^{(e)}_{(t)}\right)^2 + \int \mathcal{B}\left(\dot\phi + A^{(e)}(t)\right) dt$$

$$(*)\begin{cases} \phi(t) \longrightarrow \phi(t) + \alpha(t) \\ A^{(e)}(t) \longrightarrow A^{(e)}(t) - \partial_t \alpha(t) \end{cases}$$

Then $\dot\phi + A^{(e)}(t)$ inut.

<u>Remk</u>: $A^{(e)}(t)\,dt$ 1-form on principal G w/ connection

$P \twoheadrightarrow M$ $\quad M = $ 1-manifold of time

Relative to a local trivial the connection defines 1-form

$$A^{(e)} = A^{(e)}(t)\, dt.$$

(*) is equiv. to (**)

$$\Phi(t) \longrightarrow e^{i\alpha(t)}\,\Phi(t)$$

$$d + i\,A^{(e)} \longrightarrow e^{i\alpha(t)}\left(d + iA^{(e)}\right)e^{-i\alpha(t)}$$

(**) is better than (*)

- more geometrical
- makes sense on nontrivial

Spacetimes even when $\Phi(t)$ S.V. but $\phi(t)$ is not.

Charge Conj.

$$\phi(t) \longrightarrow -\phi(t) \qquad \Phi \to \Phi^*$$

$$A^{(e)}(t) \longrightarrow -A^{(e)}(t)$$

If we want inv't. action:

$$\mathcal{B} \longrightarrow -\mathcal{B}$$

Can show: With suitable definition $H_{\mathcal{B}}$.

[$H_{\mathcal{B}} \not\simeq H_{-\mathcal{B}}$ unless $2\mathcal{B} \in \mathbb{Z}$

still acts on $L^2(S^1) \ni \Psi(\phi)$

Important: Periodicity in $\mathcal{B}$. With $A^{(e)}(t)$ we're lost that

We can restore periodicity in $\mathcal{B}$ by adding a new term to the action:

Cov. deriv.

$$e^{-S'_{Eucl.}} = e^{-\int \frac{1}{2} I \left(\dot{\phi} + A^{(e)}\right)^2}$$

gauge invariance ?

$$\underbrace{e^{-i \oint \mathcal{B} \left(\dot{\phi} + A^{(e)}\right) dt}}_{\theta\text{-term}} \quad \underbrace{e^{ik \int A^{(e)}(t)\, dt}}_{\substack{\text{Chern-Simons} \\ \text{term} \quad k = \text{"level"}}}$$

Why not?

$$\exp \int Li_2 \left(\sin A^{(e)}(t)\right) dt$$

What about gauge invce

$$A^{(e)}(t) \longrightarrow A^{(e)}(t) - \partial_t \alpha(t)$$

arbitrary

$$\dot{\phi} + A^{(e)}(t) \to \Phi^{-1}\underbrace{\left(-i\frac{d}{dt} + A^{(e)}\right)}_{\text{Covariant derivative.}}\Phi$$

$$\int dt_1\, dt_2\, dt_3 \left(\frac{\phi(t_1) - \phi(t_2)}{\phi(t_3)}\right)^2$$

Physical Criteria for Good Actions

1. Locality
2. Unitarity
3. Gauge invariance
4. Renormalizability
5. Invariance under Symmetries of the problem

$$\int dt \; \underbrace{Li_2\left(\exp t^2\right)}\,\dddot{\phi}^{\,2}(t)$$

spoils time translation invce

$$\int \Big(\partial_\mu \phi \, \partial^\mu \phi + m^2 \phi^2 + \underline{V(\phi)} \Big) d^d x$$

~~Poincaré~~ Crystallographic Symmetry

$$\partial_x \phi \, \partial_y \phi \leftarrow$$

$$A_{\mu\nu} \, \partial^\mu \phi \, \partial^\nu \phi$$

$$\int \theta \cdot \dot{\phi}$$

YM :

$$\left.\begin{aligned} &\int \frac{1}{g^2} \operatorname{tr}(F \wedge * F) \\ + &\int \theta \operatorname{tr}(F \wedge F) \end{aligned}\right\}$$

$$\cancel{\int \left(\operatorname{tr} F \wedge * F\right)^2}$$

Polynóm in $\quad D_{\mu_1} \cdots D_{\mu_k} F_{\mu\nu} \quad \leftarrow$ local gauge cov.

Contract indices Lorentz

power counting

Back to QM problem: Adding C-S term is good because we restored — in a sense — the periodicity in $\mathcal{B}$

<u>Classically</u> $(\mathcal{B}, k) \to (\mathcal{B}+r, k+r)$

$r \in \mathbb{R}$

Q.M. $2\mathcal{B} \in \mathbb{Z}$

Charge Conj. Invce

$$(\mathcal{B}, k) \sim (-\mathcal{B}, -k)$$

$$2\mathcal{B} = 2k = N$$

Can only hope for C.C. invce $\mathcal{B} = k \in \mathbb{Z}/2$